LINEAR SYSTEMS ON GENERIC $K3$ SURFACES

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Abstract. In this paper we prove the equivalence of two conjectures on linear systems through fat points on a generic $K3$ surface. The first conjecture is exactly as Segre conjecture on the projective plane. Whereas the second characterizes such linear system and can be compared to the Gimigliano-Harbourne-Hirschowitz conjecture.

1. Introduction

In this paper we assume the ground field is algebraically closed of characteristic 0.

With S we always denote a smooth projective *generic* $K3$ surface, i.e. $\mathrm{Pic}\,(S) \cong \mathbb{Z}$.

Consider r points in general position on S, to each one of them associate a natural number m_i called the *multiplicity* of the point. Let r_j be the number of p_i with multiplicity m_i and let H be the generator of $\mathrm{Pic}\, S$.

For a linear system of curves in $|dH|$ with r_j general base points of multiplicity m_j for $j = 1 \cdots k$, define its *virtual dimension* v as $\dim |dH| - \sum r_i m_i(m_i + 1)/2$ and its *expected dimension* by $e = \max\{v, -1\}$. If the dimension of the linear system is l, then $v \leq e \leq l$.

Observe that it is possible to have $e < l$, since the conditions imposed by the points may be dependent. In this case we say that the system is *special*.

Linear systems through general fat points on rational surfaces have been studied by many authors (see e.g. [Seg62, Hir89, Gim89, Har86, CM01, CM98]), but, as far as we know, no conjecture concerning the structure of such systems on $K3$ surfaces has been formulated.

Inspired by the article [CM01] by C. Ciliberto and R. Miranda, we start with a Segre-like conjecture (conjecture 2.1), and deduce conjecture 2.3, which can be seen as a translation of the Gimigliano-Harbourne-Hirschowitz conjecture [CM01, Conjecture 3.1] to $K3$ surfaces.

In section 2 we introduce some notation and definitions and state the two conjectures. In the following section we prove that the two conjectures are in fact

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equivalent. Finally, in section 4, we prove some results which are in favor of conjecture 2.1.

2. Preliminaries

Let S be a generic $K3$ surface and let H be the generator of $\mathrm{Pic}\,(S)$, then H is ample, $H^2 = 2g-2 \geq 2$ and $h^0(H) = g+1$; moreover H is very ample if $g \geq 3$ and if $g = 2$ H defines a double covering of $\mathbb{P}^2$ branched at an irreducible sextic (see [May72, Proposition 3]).

Consider $p_1, \cdots, p_r$ points in general position on S, for each one of these points fix a multiplicity $m_1, \cdots m_r$. With $\mathcal{L} = \mathcal{L}^n(d, m_1, \cdots, m_r)$ we mean the linear system of curves in $|dH|$ with multiplicity m_i at p_i for all i, where $n = 2g-2$.

Let $Z = \sum m_i p_i$ be the 0-dimensional scheme defined by the multiple points and consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_S(dH) \otimes \mathcal{I}(Z) \longrightarrow \mathcal{O}_S(dH) \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where $\mathcal{I}(Z)$ is the ideal sheaf of Z. Taking cohomology we obtain

$$v = h^0(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - h^1(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - 1, \tag{2.1}$$

because $h^1(\mathcal{O}_S(dH)) = 0$ (see e.g. [May72]).

Equation (2.1) shows that a non-empty linear system $\mathcal{L}$ is special if and only if $h^1(\mathcal{L}) = 0$. We can now formulate the following

Conjecture 2.1. *If $\mathcal{L}$ on S is non-empty and reduced, then it is non-special.*

Remark 2.2. Conjecture 2.1 implies that a general element of a special system is reduced, which is equivalent to saying that there exists a curve C such that $2C \subseteq \mathrm{Bs}\,\mathcal{L}$ (because of Bertini's first theorem as stated in [Kle98, Theorem (4.1)]).

Conjecture 2.3. *Let $\mathcal{L}$ and S be as above, then*

(i) *$\mathcal{L}$ is special if and only if $\mathcal{L} = \mathcal{L}^4(d, 2d)$ or $\mathcal{L} = \mathcal{L}^2(d, d^2)$ with $d \geq 2$;*
(ii) *if $\mathcal{L}$ is non-empty then its general divisor has exactly the imposed multiplicities in the points p_i;*
(iii) *if $\mathcal{L}$ is non-special and has a fixed irreducible component C then*
  a) *$\mathcal{L} = \mathcal{L}^2(m+1, m+1, m) = mC + \mathcal{L}^2(1,1)$ with $C = \mathcal{L}^2(1, 1^2)$ or*
  b) *$\mathcal{L} = 2C$ with $C \in \{\mathcal{L}^4(1, 1^3), \mathcal{L}^6(1, 2, 1), \mathcal{L}^{10}(1, 3)\}$ or*
  c) *$\mathcal{L} = C$.*
(iv) *if $\mathcal{L}$ has no fixed components then either its general element is irreducible or $\mathcal{L} = \mathcal{L}^2(2, 2)$.*

3. The equivalence of the two conjectures

It is clear that conjecture 2.3 implies conjecture 2.1, and we will now show that actually they are equivalent.

For the rest of this section we assume that conjecture 2.1 is true.

If $\mathcal{L}$ and S are as above, let S' denote the blowing-up of S along the points $p_1, \ldots, p_r$, and let E_i be the exceptional divisor on S' corresponding to p_i. Then the canonical

class $K_{S'}$ of S' is equal to $\sum_{i=1}^r E_i$ and $\operatorname{Pic} S'$ is generated by $\{H, E_1, \ldots, E_r\}$, where, by abuse of notation, H also denotes the pullback of H on S'.

Let D be a divisor on S', such that $|D| = |tH - \sum_{i=1}^r l_i E_i|$ with $t \geq 0$ for all i. Then $\chi(D) = \frac{1}{2}(D^2 - DK_{S'}) + 2$, and we define the virtual dimension of D as

$$v(D) = \chi(D) - h^2(D) - 1 = h^0(D) - h^1(D) - 1.$$

By duality, $h^2(D) = 0$, unless $|D| = |\sum_{i\in I} E_i|$, with $I \subseteq \{1, \ldots, r\}$. Also note that if $t > 0$ then $|D|$ corresponds to the system $\mathcal{L}^n(t, l_1, \ldots, l_r)$ on S and $v(D) = v(\mathcal{L}^n(t, l_1, \ldots, l_r))$ (see equation (2.1)). By abuse of notation we then also denote $|D|$ by $\mathcal{L}^n(t, l_1, \ldots, l_r)$. Moreover, if C and C' are two curves on S, then by CC' we mean the intersection multiplicity of their strict transforms on S'.

Lemma 3.1. *Let $\mathcal{M}$ be a linear system on S without fixed components then either its general element is irreducible or $\mathcal{M} = \mathcal{L}^2(2, 2)$.*

Proof. Because of Bertini second theorem as stated in [Kle98, 5.3] the general element of $\mathcal{M}$ is reducible if and only it is composite with a pencil $\mathcal{P}$. Let M and P be the strict transforms on S' of general elements of $\mathcal{M}$ and $\mathcal{P}$ respectively. This means that $|M| = |lP|$ with $l = \dim |M| \geq 2$. By remark 2.2 we can say that $|M|$ and $|P|$ are non-special which gives $v(M) = l$ and $v(P) = 1$. The second equality implies that $PK_{S'} = P^2$ so the first is equivalent to $P^2 = 2/l$ which gives $l = 2$ and $P^2 = PK_{S'} = 1$. This means that $P = dH - E_i$ and $1 = P^2 = nd^2 - 1$ which is only possible if $n = 1$ and $d = 1$, i.e. $\mathcal{M} = \mathcal{L}^2(2, 2)$. □

Observe that for any divisors A, B on S' we have $\chi(A+B) = \chi(A) + \chi(B) + AB - 2$, hence if $h^2(A) = h^2(B) = 0$ then

$$v(A + B) = v(A) + v(B) + AB - 1. \tag{3.1}$$

Lemma 3.2. *Let $|D|$ be the linear system on S' corresponding to a linear system $\mathcal{L} = \mathcal{L}^n(d, m_1, \ldots, m_r)$ on S. Then $E_i \not\subseteq \operatorname{Bs}|D|$ for all $i = 1, \ldots, r$.*

Proof. Assume the statement is false, then we can write $|D| = \sum l_i E_i + F + |D'|$, with $E_i \not\subseteq \operatorname{Bs}|D'|$, F the strict transform of the fixed components of $\mathcal{L}$ and $|D'|$ without fixed components.
Assume there exists an i such that $l_i > 0$, then $E_i(D' + F) > 0$ (otherwise $m_i = DE_i = -l_i < 0$).
Assume that $E_i F > 0$. Let $C \subseteq F$ be an irreducible divisor with $CE_i > 0$. By conjecture 2.1 $\dim C = v(C) = 0$, and $C + E_i$ still has dimension 0, because it is contained in $\operatorname{Bs}|D|$. On the other hand, $C + E_i$ is non-special (since C is non-special), so $v(C + E_i) = v(C)$. This implies $\chi(C) = \chi(C + E_i)$ which is equivalent to $CE_i = 0$ and thus contradicts our assumption.
So we get that $E_i D' > 0$. By conjecture 2.1 $\dim |D'| = v(D')$, and, as before, $|D'| + E_i$ is non-special. So $D'E_i = 0$ which again contradicts our assumption. □

Lemma 3.3. *If C is an irreducible divisor on S' such that $v(C) = 0$ and $C^2 \leq 1$, then C is one of the following:*

C^2	C		
≤ -1	$\emptyset$		
0	$\mathcal{L}^4(1,2)$	$\mathcal{L}^2(1,1^2)$	
1	$\mathcal{L}^4(1,1^3)$	$\mathcal{L}^6(1,2,1)$	$\mathcal{L}^{10}(1,3)$

Proof. If $h^2(C) > 0$, then $C = E_i$ for some i, in which case $v(C) = 0$ and $C^2 = -1$. If $h^2(C) = 0$, then $v(C) = 0$ implies that $CK_{S'} = C^2 + 2$. But $C^2 + 2 = p_a(C) \geq 0$, so $C^2 \geq -2$.
In case $C^2 \leq -1$, $CK_{S'} \leq 1$ so $C = tH - aE_i$ with $a \in \{0, 1\}$, which gives $t^2 n - a \leq -1$ and this is not possible if $t > 0$.
In case $C^2 = 0$ then $CK_{S'} = 2$, so either $C = tH - 2E_i$ or $C = tH - E_i - E_j$. In the first case, $C^2 = nt^2 - 4 = 0$, which is only possible if $n = 4$ and $t = 1$. In the latter case, $C^2 = nt^2 - 2 = 0$, which is only possible if $n = 2$ and $t = 1$.
In case $C^2 = 1$ then $CK_{S'} = 3$, so C is of the following types $tH - 3E_i$, $tH - 2E_i - E_j$ or $tH - E_i - E_j - E_k$. In the first case, $C^2 = nt^2 - 9 = 1$, which is only possible if $n = 10$ and $t = 1$. In the second case, $C^2 = nt^2 - 5 = 1$, which is only possible if $n = 6$ and $t = 1$. And in the latter case, $C^2 = nt^2 - 3 = 1$, which is only possible if $n = 4$ and $t = 1$. □

Proposition 3.4. *Let $\mathcal{L} = \mathcal{L}^n(d, m_1, \ldots, m_r)$ be a linear system on S and assume that there exist distinct irreducible curves C_i and D_j such that the fixed part of $\mathcal{L}$ is given by $\sum_{i=1}^a \mu_i C_i + \sum_{i=1}^b D_i$, where $\mu_i \geq 2$, then either $\dim \mathcal{L} = 0$ and it is one of the following:*

mC	$C^2 = 0$	$v(C) = 0$	*special for $m \geq 2$ with $v(\mathcal{L}) = 1 - m$*
$2C$	$C^2 = 1$	$v(C) = 0$	*non-special*
D	$D^2 \geq 0$	$v(D) = 0$	*non-special*

or $\mathcal{L} = \mathcal{L}^2(m+1, m+1, m) = m\mathcal{L}^2(1, 1^2) + \mathcal{L}^2(1, 1)$ which is non-special. Note that in the latter case $m\mathcal{L}^2(1, 1^2)$ is the fixed part of $\mathcal{L}$ and $\mathcal{L}^2(1, 1)$ is its free part of dimension 1.

To simplify the proof of this proposition, we first give two lemmas.

Lemma 3.5. *With the same situation of proposition 3.4 we have $C_iC_j = C_iD_j = D_iD_j = 1$ and $C_i^2 \leq 1$.*

Proof. Since $C_i + C_j \subset \mathrm{Bs}(\mathcal{L})$ then $\dim |C_i + C_j| = 0$ and conjecture 2.1 implies that $v(C_i + C_j) = 0$. Using the same argument one also obtains that $v(C_i) = v(C_j) = 0$, so equation (3.1) implies that $C_iC_j = 1$. In the same way one can proof that $C_iD_j = 1$ and $D_iD_j = 1$.

From $v(C_i) = 0$ we obtain $C_iK_{S'} = C_i^2 + 2$; since $2C_i \subset \mathrm{Bs}(\mathcal{L})$ this implies that $v(2C_i) \leq 0$ which gives $C_i^2 \leq 1$. □

Lemma 3.6. *Let A and B be the strict transforms on S' of two distinct irreducible curves on S then one of the following holds:*

$A = \mathcal{L}^2(1, 1^2)$ and B is an irreducible element of $\mathcal{L}^2(1, 1)$; or
$AB \neq 1$.

Proof. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{S'}(A-B) \longrightarrow \mathcal{O}_{S'}(A) \longrightarrow \mathcal{O}_B(A) \longrightarrow 0.$$

Using conjecture 2.1, we obtain $h^1(\mathcal{O}_{S'}(A)) = 0$ and as $h^2(\mathcal{O}_{S'}(A)) = 0$ the preceding sequence implies that $h^2(\mathcal{O}_{S'}(A-B)) = h^1(\mathcal{O}_B(A))$. Moreover, by Riemann-Roch and $h^0(\mathcal{O}_B(A)) = 1$ we obtain that $h^1(\mathcal{O}_B(A)) = p_a(B) - 1$.

Observe that $p_a(B) = (B^2 + BK_{S'})/2 + 1 \geq 2$; indeed, $B^2 \geq 0$ and $BK_{S'} \geq 0$ but one can immediately verify that they can not be both 0.

The preceding calculation shows that, by Serre duality:

$$\dim |K_{S'} + B - A| \geq p_a(B) - 2.$$

By interchanging the roles of A and B we obtain also $\dim |K_{S'} + A - B| \geq p_a(A) - 2$. This implies that $\dim |2K_{S'}| \geq (p_a(A) - 2) + (p_a(B) - 2)|$. Since $\dim |2K_{S'}| = 0$ this means that $p_a(A) = p_a(B) = 2$.

From the exact sequence we also see that

$$h^0(\mathcal{O}_{S'}(A)) \leq h^0(\mathcal{O}_{S'}(A-B)) + h^0(\mathcal{O}_B(A));$$

on the other hand if $h^0(\mathcal{O}_{S'}(A-B)) \geq 2$ then $\dim |K_{S'}| \geq \dim |K_{S'} + B - A| + \dim |A - B| \geq 1$ which is not possible. So this implies that $\dim |A| \leq 1$.

We already know that $p_a(A) = 2$ and the preceding analysis shows that $0 \leq v(A) \leq 1$. In case $v(A) = 0$ we obtain $A^2 = 0$ so by lemma 3.3 $A = \mathcal{L}^4(1,2)$ or $A = \mathcal{L}^2(1,1^2)$. Otherwise $v(A) = 1$ and $A^2 = 1$, this gives $AK_{S'} = 1$ hence the only possibility is given by $A = \mathcal{L}^2(1,1)$.

The same considerations are also true for B, hence the only possible pair A, B such that $AB = 1$ is given by: $A = \mathcal{L}^2(1,1^2)$ and B is an irreducible element of $\mathcal{L}^2(1,1)$. □

Proof of proposition 3.4. Let $|L|$ be the linear system on S' with L the strict transform of a general element of $\mathcal{L}$ on S. Then

$$|L| = \sum_{i=1}^{a} \mu_i C_i + \sum_{i=1}^{b} D_i + |M|,$$

where $|M|$ is without fixed components and C_i and D_j are distinct irreducible curves. And, by lemma 3.2, $\sum_{i=1}^{a} \mu_i C_i + \sum_{i=1}^{b} D_i$ is the strict transform of the fixed part of $\mathcal{L}$.
Because of lemmas 3.5 and 3.6 $a + b = 1$.
Assume that $|M|$ is non-trivial, i.e. $\dim \mathcal{L} > 0$. Let C be an irreducible fixed component of $|L|$, then, by conjecture 2.1, $v(M) = v(M + C)$, which implies that $MC = 1$. By lemma 3.1 either the general element of $|M|$ is irreducible or $|M| = \mathcal{L}^2(2,2)$.
In the latter case a general element of $|M|$ can be written as $M_1 + M_2$ with $M_i \in \mathcal{L}^2(1,1)$, so MC cannot be equal to 1 as it is always even.
In the first case, by lemma 3.6, the only possibility is $C = \mathcal{L}^2(1,1^2)$ and $|M| = \mathcal{L}^2(1,1)$, i.e. $\mathcal{L} = \mathcal{L}^2(m+1, m+1, m) = m\mathcal{L}^2(1,1^2) + \mathcal{L}^2(1,1)$. In order to see that the last equality is true, we just have to note that $\dim \mathcal{L} = 1$. Indeed, by

specializing the two general points of S to points on the ramification divisor, the obtained system corresponds to $\mathcal{O}_{\mathbb{P}^2}(m+1) \otimes \mathcal{I}(Z)$ with $Z = (m+1)p_1 + mp_2$, which, obviously has dimension 1.
Now assume that $\dim \mathcal{L} = 0$. In case $b = 1$, by conjecture 2.1, $v(D) = 0$, and, by lemma 3.3, we know that $D^2 \geq 0$. In case $a = 1$, by lemma 3.5 we know that $\mathcal{L} = mC$ with $C^2 \leq 1$ and $v(C) = 0$ (because of conjecture 2.1). If $C^2 = 0$ then $CK_{S'} = 2$, so $v(mC) = 1-m$. If $C^2 = 1$ then $CK_{S'} = 3$, so $v(mC) = m(m-3)/2+1$ thus $v(mC) \leq 0$ implies $m = 2$. □

Note that proposition 3.4 and lemma 3.3 imply (i) and (iii) of conjecture 2.3, part (ii) follows from lemma 3.2 and lemma 3.1 implies (iv); so we proved the following

Theorem 3.7. *Conjecture 2.1 implies conjecture 2.3.*

4. Results in favor of conjecture 2.1

In this section we will list some results which leads us to believe that conjecture 2.1 is true.

Theorem 4.1. *Let $\mathcal{L}$ be a non-special linear system on a smooth projective surface X such that $\mathcal{L} \otimes \mathcal{I}(2p)$ is special for a general point $p \in X$, then $\mathcal{L} \otimes \mathcal{I}(2p)$ has a double fixed component through p.*

Proof. We may assume that $\mathcal{L}$ has no fixed components, because otherwise we can consider $\mathcal{L} - F$, where F is the fixed divisor of $\mathcal{L}$. Let $n = \dim \mathcal{L}$ and consider the rational map $\varphi : X \dashrightarrow \mathbb{P}^n$ corresponding to $\mathcal{L}$. Saying that $\mathcal{L} \otimes \mathcal{I}(2p)$ is special for a general point $p \in X$, means that the image $X' = \varphi(X)$ has to be a curve. Indeed, the speciality implies that an hyperplane which contains $p' = \varphi(p)$ and one tangent direction $\tau \in T_{p'}(X')$ has to contain the whole tangent space $T_{p'}(X')$. Because $\mathcal{L}$ is given by $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)|_{X'})$, the general divisor of $\mathcal{L}$ can be written as $\varphi^*(\sum n_i p_i) = \sum n_i F_i$, with $\sum n_i = \deg X'$. So any divisor of $\mathcal{L} \otimes \mathcal{I}(2p)$ contains $2F_p$ with $F_p = \varphi^*(\varphi(p))$, i.e. $2F_p \subset \mathrm{Bs}\,(\mathcal{L} \otimes \mathcal{I}(2p))$. □

If it is possible to find reduced curves C_1, C_2 on a generic $K3$ surface such that C_1 and C_2 have no common components, $v(C_1) = v(C_2) = 0$ and $v(C_1 + C_2) < 0$, then this would imply that conjecture 2.1 is false because $C_1 + C_2$ would be a special system with no multiple components. We will show that such curves can not exist in the following

Proposition 4.2. *Let S be a generic $K3$ surface with $H^2 = n$, let C and C' be curves on S with $v(C) = v(C') = 0$, then $v(C + C') \geq 0$ unless $C = C' \in \{\mathcal{L}^2(1, 1^2), \mathcal{L}^4(1, 2)\}$.*

Proof. Let $C \in \mathcal{L}^n(d, m_1, \dots, m_r)$ and $C' \in \mathcal{L}^n(d', m'_1, \dots, m'_r)$. Then $v(C) = 0$, resp. $v(C') = 0$, implies that $nd^2 = \sum_{i=1}^r m_i(m_i+1) - 2$, resp. $nd'^2 = \sum_{i=1}^r m'_i(m'_i+1) - 2$, so

$$(ndd')^2 = (\sum_{i=1}^r m_i(m_i+1) - 2)(\sum_{i=1}^r m'_i(m'_i+1) - 2).$$

Moreover $v(C+C') \geq 0$ is equivalent to $ndd' > \sum_{i=1}^r m_i m'_i$, which gives

$$(\sum_{i=1}^r m_i(m_i+1)-2)(\sum_{i=1}^r m'_i(m'_i+1)-2) > (\sum_{i=1}^r m_i m'_i)^2. \tag{4.1}$$

Note that, by Schwartz inequality,

$$(\sum m_i^2)(\sum {m'}_i^2) - (\sum m_i m'_i)^2 \geq 0,$$

and we denote the left hand side by t. To simplify notation we also denote $a = \sum m_i^2$, $a' = \sum {m'}_i^2$, $b = \sum m_i$ and $b' = \sum m'_i$. Then the inequality (4.1) is equivalent to

$$t + (a-2)(b+b'-2) + a'(b-2) > 0.$$

Because $v(C) = v(C') = 0$, we know that $a, b \geq 2$, so the preceding inequality is true unless $a = b = 2$ and $t = 0$. We know that $t = 0$ if and only if there exists a constant c such that $m'_i = cm_i$ for all i. On the other hand $a = b = 2$ implies that $C, C' \in \{\mathcal{L}^n(d, 1^2), \mathcal{L}^n(d, 2)\}$, and $0 = v(\mathcal{L}^n(d, 1^2)) = nd^2/2 - 1$ is only possible for $n = 2$ and $d = 1$, while $0 = v(\mathcal{L}^n(d, 2)) = nd^2/2 - 2$ is only possible for $n = 4$ and $d = 1$. □

Corollary 4.3. *Lemma 3.5 is true without assuming conjecture 2.1.* □

Corollary 4.4. *With the same assumptions as in proposition 3.4 we have that $a + b \leq 2$ (without assuming conjecture 2.1).*

Proof. Assume that $a+b \geq 3$ and let $R_1, R_2, R_3 \subseteq \operatorname{Bs} \mathcal{L}$ be three distinct irreducible curves. Then $v(R_i) = 0$ and, since $R_i + R_j \subseteq \operatorname{Bs} \mathcal{L}$, proposition 4.2 implies that $v(R_i + R_j) = 0$; which in turn is equivalent to $R_i R_j = 1$. But, in the same way we can prove that $R_1(R_2 + R_3) = 1$, which contradicts $R_i R_j = 1$. □

Remark 4.5. Observe that lemma 3.3 is true without assuming conjecture 2.1; and the same is true for lemma 3.1, with just minor changes to the proof.

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